

Method of images applied to an opto-mechanical Fabry-Perot resonator

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We show here that effectively relativistic motions of the images of objects in a Fabry-Perot resonator can occur. Since these images can move relativistically, they can emit radiation that can be seen by a stationary observer (e.g., the eye in Figure 1).

Consider the opto-mechanical Fabry-Perot resonator operating at microwave frequencies sketched in Figure 1. Let the axis of the resonator be the z axis, and let the mass of stationary mirror M1 be a large mass M , but let the mass of the moving mirror M2 be a small, mesoscopic mass m . For example, M2 could be a Planck-scale mass of a thin, flexible superconducting membrane mirror that can easily be driven into motion at microwave frequencies [1][2].

The left mirror M1 of this resonator, being heavy, is essentially motionless, but its right mirror M2, being light, can be moving to the right relative to the mirror M1 with an instantaneous velocity \mathbf{v} , as seen by the eye in Figure 1. We shall assume that the left mirror M1 will be negatively charged with a charge $-q$ and that the right mirror M2 will be positively charged with a charge $+q$, like in a capacitor.

The light mirror M2 can be mechanically driven, for example, by some microwave-frequency Coulomb force $F_z(t)$ due to an instantaneous longitudinal electric field $E_z(t)$ of a transverse magnetic mode of the resonator, via the relationship [1][2]

$$F_z(t) = qE_z(t) \quad (1)$$

The two mirrors M1 and M2 could be composed of superconductors, so that the quality factor Q of the Fabry-Perot resonator could be extremely high, for example, on the order of

$$Q \sim 10^{10} \quad (2)$$

as has been demonstrated by Haroche and co-workers [3]. Let the distance between the two mirrors M1 and M2 be L , which, for the lowest mode of the resonator, will be around $\lambda/2$, i.e., half a microwave wavelength. We shall apply the method of images to solve this problem.

The magnitude of the instantaneous velocity of mirror M2 moving to the right relative to mirror M1 will be given by

$$v = \dot{L} \quad (3)$$

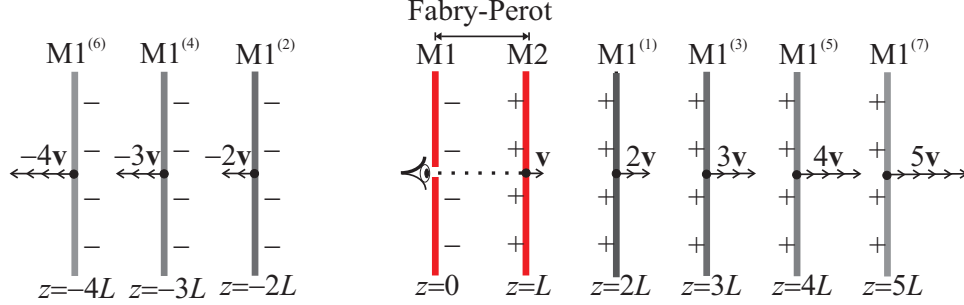


Figure 1: Fabry-Perot resonator with one stationary mirror M1 and one moving mirror M2. The observer (the eye) will see multiple images $M1^{(m)}$, ($m = 1, 2, 3, \dots$), of the mirror M1 with steadily increasing velocities as the order m of the image of M1 increases.

Using the mathematical method of induction in conjunction with the physical method of images, one finds that the positions $z^{(2n+1)}$ of successively higher odd order images $M1^{(2n+1)}$, ($n = 0, 1, 2, \dots$), of the mirror M1 on the right side of mirror M2, as seen by the eye in Figure 1, will be given by

$$z^{(2n+1)} = (n + 2)L, \quad (n = 0, 1, 2, \dots) \quad (4)$$

Therefore the magnitudes of the velocities of the successively higher-order images of mirror M1 on the right side of mirror M2 will be given by

$$v^{(2n+1)} = \dot{z}^{(2n+1)} = (n + 2)\dot{L} = (n + 2)v, \quad (n = 0, 1, 2, \dots) \quad (5)$$

One concludes that the velocity of the higher-order images to the right of M2 will steadily increase as the order n of the images $M1^{(2n+1)}$ steadily increases.

Now the meaning of the quality factor Q of the resonator is that it is roughly the number of back-and-forth reflections of a ray between mirrors M1 and M2 before the ray escapes the resonator. Hence let us set $n \sim Q$ for the total number of bounces of a ray within the Fabry-Perot configuration of Figure 1, before it leaves the cavity. A rough estimate of the magnitude of the velocity of the highest order image of M1 just before the ray escapes from the resonator, will therefore be on the order of

$$v^{(2Q+1)} = \dot{z}^{(2Q+1)} = (Q + 2)\dot{L} = (Q + 2)v \sim Qv \quad (6)$$

where $Q \sim 10^{10}$. Therefore, although the mirror M2 may initially be moving nonrelativistically with $v \ll c$, nonetheless after Q reflections within the Fabry-Perot, the effective velocity of the Q th order image of M1, namely $M1^{(2Q+1)}$, can become *effectively* relativistic.

For example, if mirror M2 initially moves with a nonrelativistic velocity $v \sim 1 \text{ cm s}^{-1}$, then after $Q \sim 10^{10}$ reflections, the Q th order image of M1 will

appear to be moving with a relativistic velocity on the order of $v^{(2Q+1)} \sim 10^{10}$ cm s⁻¹. Since the relativistically moving images of M1 will be carrying both a charge and a mass with them as they move, these images can radiate significant amounts of both electromagnetic and gravitational radiation [4].

One concludes that it is not necessary for the center of mass of quantum mesoscopic objects to move near the speed of light in order for them to emit appreciable amounts of radiation. Rather, it is sufficient for their *images* to *effectively* move relativistically inside an extremely high Q resonator, such as the SC resonator in Figure 1.

Let us estimate the order of magnitude of the gravitational wave power emitted in the quadrupolar radiation arising from the moving masses of the images of Figure 1. We shall assume at the outset that the SC “mirror” boundary conditions of [5] are valid here, which allows us then to use the method of images. But first let us estimate the amount of electromagnetic wave power emitted by the moving charges of the images of Figure 1, starting from the following formula for the emission of quadrupole radiation given by [6]

$$\mathcal{P}^{(\text{quad})} = \frac{Z_0 \omega^6}{1440 \pi c^4} \sum_{ij} |\mathcal{Q}_{ij}(\omega)|^2 \quad (7)$$

where $Z_0 = 377$ ohms is the characteristic impedance of free space, and where $\mathcal{Q}_{ij}(\omega)$ is a component of the electric quadrupolar moment tensor of a radiating source which is oscillating at an angular frequency ω . We shall apply (7) to the radiation emitted by the dominant oscillating quadrupole moment as seen by the eye in Figure 1, which will arise from the moving image charge $M1^{(2n+1)}$, relative to the moving image $M1^{(1)}$, when $n \approx Q$, the quality factor of the resonator. Now the dominant component of the electrostatic quadrupole moment $\mathcal{Q}_{zz}^{(\text{stat})}$ as seen by the eye in Figure 1 arising from the charge $+q$ on the mirror image $M1^{(2Q+1)}$, relative to the image charge $+q$ on the mirror image $M1^{(1)}$, will be given by

$$\mathcal{Q}_{zz}^{(\text{stat})} \approx q d_0^2 \quad (8)$$

where $d_0 \approx QL \approx Q\lambda/2$ is the distance from $M1^{(2Q+1)}$ to $M1^{(1)}$, and where $L = \lambda/2$ is a microwave half-wavelength. Let the motion of the mirror M2 be given by a small, sinusoidally varying displacement $\Delta d(t) = \varepsilon_{\max} \sin \omega t$ with a small displacement amplitude ε_{\max} . It follows that the time-varying component of the dominant quadrupole moment formed by the images $M1^{(1)}$ and $M1^{(2Q+1)}$ will be

$$\Delta \mathcal{Q}_{zz}(t) \approx 2qQd_0 \varepsilon_{\max} \sin \omega t \quad (9)$$

From (7), one then obtains an estimate of the time-averaged emitted power

$$\mathcal{P}^{(\text{quad})} \approx \frac{\pi}{720} \frac{Z_0 \omega^4}{c^2} q^2 Q^4 \varepsilon_{\max}^2 \quad (10)$$

as seen by the eye in Figure 1. Now the on-resonance solution to the simple harmonic oscillation equation of motion of the moving mirror M2 with a mass

m is given by

$$\varepsilon_{\max} = \frac{qE_{\max}}{m\omega^2}Q \quad (11)$$

where E_{\max} is the maximum amplitude of the longitudinal electric field being applied to the electrostatic charge $+q$ of M2 in the transverse magnetic mode. From (10) and (11), we see that the emitted power in quadrupole radiation from the images $M1^{(1)}$ and $M1^{(2Q+1)}$ will be proportional to the power injected into the resonator which is driving the motion of the mirror M2. Let \mathcal{P}_i be this injected power. This leads to the relationship

$$E_{\max}^2 = \frac{Z_0 Q}{\mathcal{A}_{\text{eff}} \pi} \mathcal{P}_i \quad (12)$$

where $\mathcal{A}_{\text{eff}} \approx \lambda^2$ is the effective cross-sectional area of the transverse magnetic mode of the resonator. One then finds that

$$\mathcal{P}^{(\text{quad})} = \eta_{\text{eff}} \mathcal{P}_i \quad \text{with} \quad \eta_{\text{eff}} \approx \frac{Z_0^2 q^4 Q^7}{720 m^2 c^2 \lambda^2} \quad (13)$$

where η_{eff} is the effective conversion efficiency from \mathcal{P}_i into $\mathcal{P}^{(\text{quad})}$. Now the maximum possible value of the effective conversion efficiency η_{eff} will be unity. Therefore the electrostatic charge q deposited onto M2 needed to achieve $\eta_{\text{eff}} \approx 1$ is therefore on the order of

$$q(\eta_{\text{eff}} \approx 1) \approx \left(\frac{720 m^2 c^2 \lambda^2}{Z_0^2 Q^7} \right)^{1/4} \quad (14)$$

which, when evaluated for the following parameters,

$$m = m_{\text{Planck}} = 21.8 \text{ micrograms} \quad (15)$$

$$\lambda = 1.5 \text{ centimeters (i.e., 20 GHz)} \quad (16)$$

$$Q \approx 10^9 \quad (17)$$

yields an order-of-magnitude value for the required electrostatic charge to be deposited onto M2 which is given by

$$q(\eta_{\text{eff}} \approx 1) \approx 1.4 \times 10^{-17} \text{ Coulombs} \approx 93 \text{ electrons} \quad (18)$$

which should be easily feasible. The resulting charge-to-mass ratio of the moving mirror M2 is

$$\frac{q(\eta_{\text{eff}} \approx 1)}{m} \approx 6 \times 10^{-10} \text{ Coulombs kg}^{-1} \quad (19)$$

which is within an order of magnitude of the “criticality” charge-to-mass ratio (20) (see [7]), which is the condition that when two identical charges q are attached to two identical masses m moving anti-symmetrically with respect to each other in sinusoidal motion, this pair of particles will radiate equal amounts of electromagnetic and gravitational wave power. This indicates that the amount of quadrupolar gravitational radiation emitted by the configuration of image

masses in Figure 1 can easily be made comparable to the amount of quadrupolar electromagnetic radiation emitted by the image masses of this configuration by adjusting the amount of electrostatic charge deposited on M2. Therefore the configuration sketched in Figure 1 could become the basis for a design of an efficient converter (i.e., transducer) from electromagnetic to gravitational radiation, and vice versa.

References

- [1] R.Y. Chiao, L.A. Martinez, S.J. Minter, A. Trubarov, “Parametric oscillation of a moving mirror driven by radiation pressure in a superconducting Fabry–Perot resonator system,” *Phys. Scr. T* **151**, 014073 (2012); arXiv: 1207.6885.
- [2] R.Y. Chiao, R.W. Haun, N.A. Inan, B.S. Kang, L.A. Martinez, S.J. Minter, G.A. Muñoz, and D.A. Singleton, “A gravitational Aharonov-Bohm effect, and its connection to parametric oscillators and gravitational radiation,” (Aharonov 80th birthday Festschrift article), arXiv: quant-ph 1301.4297. When the charge q on the mirrors is sufficiently large, and when a transverse magnetic mode of the resonator is being excited, then the motion of a thin, flexible mirror will be “slaved” to that of the longitudinal electric field of this transverse magnetic mode, so that the mirror will be forced to move at the microwave resonance frequency of this mode instead of its acoustical resonance frequency.
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- [4] Based on Thomson’s model for radiation from an accelerated charge due to a propagating “kink” superimposed upon a line of the Coulomb field of the charge that forms at the “speed-of-light circle” (see Figure 2(a)), one infers that an accelerated image charge (see Figure 2(b)), which is seen by a stationary observer as the image of a stationary charge $-q$ observed within the accelerated mirror, can emit radiation just as surely as an accelerated charge would radiate as it moves towards a stationary mirror. It is the conducting boundary conditions of the “kicked” mirror at point a on the surface of the moving mirror M in Figure 2(b) that leads to the formation of Thomson’s “kink” by the charge $+q$ of the accelerated image as seen within the accelerated mirror by the observer. Note that the original, real charge $-q$ is stationary and not moving in Figure 2 (b). Nevertheless, radiation will be produced by the time-varying surface currents induced in the “kicked” mirror by the stationary charge $-q$. By the relativity of motion, it cannot matter whether it is the mirror that is moving towards a stationary charge, or that it is the charge that is moving towards a stationary mirror. Therefore the

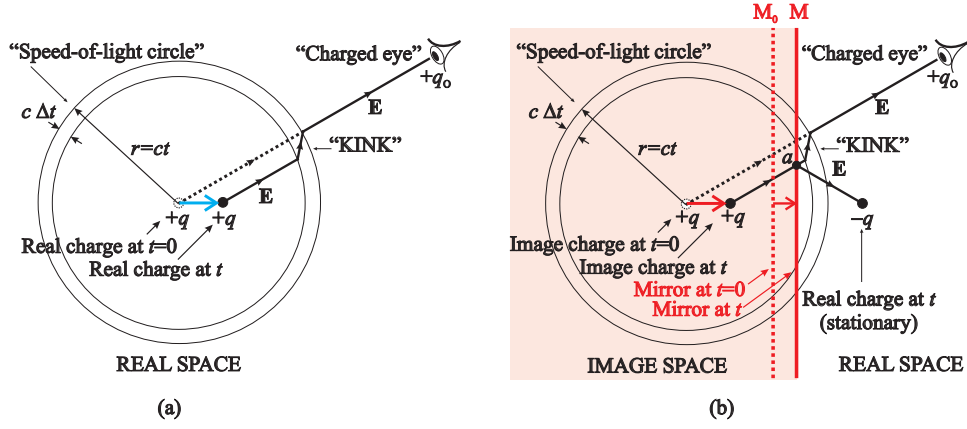


Figure 2: (a) Snapshot of a charged eye looking at a “kicked” charge. A “kink” develops in an electric field line \mathbf{E} emanating from the charge at the “speed-of-light circle,” which represents the radiation from the kicked charge. (b) Snapshot of a charged eye looking at a “kicked” mirror. Again, a “kink” develops at the “speed-of-light circle” representing radiation from the “kicked” mirror.

energy produced in the radiation associated with the “kink” by the moving mirror in the presence of the stationary charge, must be exactly the same as the energy produced in radiation associated with the “kink” by the moving charge in the presence of the stationary mirror. Since we know that energy will be radiated away in the latter case, it follows that energy must also be radiated away in the former case. Therefore accelerated image charges must radiate. Similarly, the superconducting “mirror” boundary conditions found in [5] will lead to the formation of Thomson’s “kink” by the mass of a moving image as seen within a moving mirror by a stationary observer. Therefore energy will be radiated away in the form of gravitational radiation by an accelerated SC mirror moving relative to a stationary mass, just as surely as an accelerated mass moving relative to a stationary mirror would radiate away energy in this form of radiation.

- [5] S.J. Minter, K. Wegter-McNelly, and R.Y. Chiao, “Do mirrors for gravitational waves exist?”, *Physica E* **42**, 234 (2010); arXiv:0903.0661.
- [6] J.D. Jackson, *Classical Electrodynamics*, 4th edition, (John Wiley & Sons, 1999), p.415, Eq (9.49).
- [7] If the charge-to-mass ratio of two identical charges q attached to two identical masses m , which are moving anti-symmetrically with respect to each other in sinusoidal motion, were to be adjusted to the “criticality” value [8]

$$\left(\frac{q}{m}\right)_{\text{criticality}} = \sqrt{4\pi\epsilon_0 G} = 8.6 \times 10^{-11} \text{ C/kg} \quad (20)$$

then equal amounts of electromagnetic and gravitational radiation would be produced by the sinusoidal motion of this pair of particles (see text above). Thus an efficient transducer between gravitational and electromagnetic radiation could be constructed.

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